APPLICATION OF THE METHOD OF ASSOCIATED FIELDS TO SPATIAL SELF-CONTAINED SYSTEMS

PMM Vol. 32, No. 1, 1968, pp. 131-135

I.M. BELEN'KII
(Moscow)

(*Received July 26, 1967*)

The method of associated (adjoint) fields for two-dimensional self-contained systems and its application to nonlinear vibrations of systems with one degree of freedom [1 and 2] are extended to more general three-dimensional self-contained systems, when the motion of the image point of the system takes place in three-dimensional phase space.

As an example, the interaction of a self-oscillatory system with an energy source which maintains these oscillations [3] is considered.

1. We consider a three-dimensional self-contained system of the form

$$x^{\cdot} = p(x, y, z), \qquad y^{\cdot} = q(x, y, z), \qquad z^{\cdot} = r(x, y, z) \tag{1.1}$$

Here x, y, z are the coordinates of the phase point M in Euclidean space E_3, and the functions p, q, and r are assumed to be of class C_2 in their domain of definition.

To the velocity field $\mathbf{v} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ of the phase point $M(x, y, z)$ there corresponds an associated force field [1 and 2] of the form $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where the components $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are equal, respectively, to $x^{\cdot\cdot}$, $y^{\cdot\cdot}$, and $z^{\cdot\cdot}$, and hence by virtue of system (1.1) may be put into the form

$$P = (\operatorname{grad} p \cdot \mathbf{v}), \qquad Q = (\operatorname{grad} q \cdot \mathbf{v}), \qquad R = (\operatorname{grad} r \cdot \mathbf{v}) \tag{1.2}$$

2. The following Theorems hold.

T h e o r e m 2.1. Any associated field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ may be normalized and represented in the form of a superposition of two fields, of which one is a potential field and the other a field of gyroscopic forces.

In order to see this we introduce two vector fields into consideration: one is the gradient of the function $V = -\frac{1}{2}(p^2 + q^2 + r^2)$, and the other is $\Gamma = (\operatorname{rot} \mathbf{v} \times \mathbf{v})$.

Then from (1.2) we easily obtain

$$\mathbf{F} = -\operatorname{grad} V + \Gamma \qquad (\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \tag{2.1}$$

which proves the assertion, since Γ is a gyroscopic field, and the work done by it in a displacement $d\mathbf{s}\,(dx, dy, dz)$ is equal to $(\operatorname{rot} \mathbf{v} \times \mathbf{v}) \cdot d\mathbf{s} = 0$.

The representation of the associated field $\mathbf{F}$ in the form (2.1) (the theorem on normalization) is analogous to the Gromeko-Lamb transformation in hydrodynamics [4].

We denote $\operatorname{rot} \mathbf{v}$ by $\Omega(\xi, \eta, \zeta)$. Then the associated equations of motion

$$x^{\cdot\cdot} = P(x, y, z), \qquad y^{\cdot\cdot} = Q(x, y, z), \qquad z^{\cdot\cdot} = R(x, y, z) \tag{2.2}$$

may, because of the above theorem, be put into the following form:

$$x^{\cdot\cdot} + \zeta y^{\cdot} - \eta z^{\cdot} = -\frac{\partial V}{\partial x}, \quad y^{\cdot\cdot} + \xi z^{\cdot} - \zeta x^{\cdot} = -\frac{\partial V}{\partial y}, \quad z^{\cdot\cdot} + \eta x^{\cdot} - \xi y^{\cdot} = -\frac{\partial V}{\partial z} \tag{2.3}$$

These equations describe the motion of a Lagrange system with Lagrangian

$$L = {}^1/_2\,(x'^2 + y'^2 + z'^2) - px' - qy' - rz' - V \quad (V = -{}^1/_2\,(p^2 + q^2 + r^2))$$

and are generalizations of the equations of Birkhoff [5] obtained for two-dimensional systems.

T h e o r e m 2.2. For the field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ associated with system (1.1) to be conservative, it is necessary and sufficient that

$$\operatorname{rot}(\Omega \times \mathbf{v}) = 0 \qquad (\Omega = \operatorname{rot} \mathbf{v}) \tag{2.4}$$

This is a consequence of Theorem 2.1 and the condition rot $\mathbf{F} = 0$.

C o r o l l a r y 2.1. If the associated force field is conservative and therefore satisfies (2.4), then there necessarily exists a function $\psi(x, y, z)$ such that

$$\operatorname{grad} \psi(x, y, z) = (\Omega \times \mathbf{v}) \tag{2.5}$$

The potential $V^*(x, y, z)$ of the field $\mathbf{F}$ will in this case by virtue of (2.1) be equal to

$$V^* = V - \psi \qquad (V = -{}^1/_2\,(p^2 + q^2 + r^2)) \tag{2.6}$$

C o r o l l a r y 2.2. The function $\psi(x, y, z)$ satisfies Poisson's equation

$$\Delta\psi(x, y, z) = \mu(x, y, z) \qquad (\mu = \mathbf{v}\cdot\operatorname{rot}\Omega - \Omega^2) \tag{2.7}$$

This follows immediately upon taking the divergence of both sides of (2.5).

C o r o l l a r y 2.3. If one excludes the trivial case $\mathbf{v} = 0$, then according to (2.5), ψ is identically constant when:

a) rot $\mathbf{v} = 0$, i.e. the velocity field of the point in phase space is a potential field. In this case the Pfaffian form $(pdx + qdy + rdz)$ will be a total differential, and consequently the system (1.1) belongs to the class of so-called potential systems [6];

b) the vectors $\mathbf{v}$ and rot $\mathbf{v}$ are collinear. This leads to the relation

$$\frac{r_y - q_z}{p} = \frac{p_z - r_x}{q} = \frac{q_x - p_y}{r}$$

There is no analog of case (b) for the two-dimensional systems since for plane-parallel motion rot $\mathbf{v} \perp \mathbf{v}$.

C o r o l l a r y 2.4. If rot $\mathbf{v}$ is a constant vector, then the condition (2.4) for conservativeness reduces to the relation

$$(\operatorname{div} \mathbf{v})\,\Omega_0 = (\Omega_0 \cdot \nabla)\,\mathbf{v} \qquad (\operatorname{rot} \mathbf{v} = \Omega_0) \tag{2.8}$$

In this case the function $\psi(x, y, z)$ by virtue of (2.7) will satisfy Poisson's Eq.

$$\Delta\psi = -\Omega_0^2$$

C o r o l l a r y 2.5. For two-dimensional self-contained systems ($z = 0$) and condition of conservativeness (2.4) may be reduced to the form

$$\frac{\partial}{\partial x}(p\zeta) + \frac{\partial}{\partial y}(q\zeta) = 0 \qquad (\zeta = q_x - p_y) \tag{2.9}$$

which coincides with a result obtained earlier [2]. The case ζ = const yields div $\mathbf{v} = 0$.

T h e o r e m 2.3. The function $\psi(x, y, z)$ given by (2.5) is an integral of the associated system (2.3) and represents a Hamiltonian H taken with opposite sign.

This follows immediately if one writes down the generalized energy integral and uses a Lagrangian $L = T - V^*$, where V^* is defined by (2.6).

Thus $\psi(x, y, z)$ = const is the equation of phase surfaces for the system (1.1).

T h e o r e m 2.4. If the associated equations of motion (2.3) admit two first integrals

$$F_i(x, y, z, x', y', z') = \text{const} \qquad (i = 1,2) \tag{2.10}$$

independent of (1.1), then the equations of the phase trajectories may be obtained by simple substitution.

In fact, since the phase trajectories of (1.1) are a subset of the set of trajectories of the associated system (2.3), it follows that they also satisfy (2.10). Hence by virtue of the independence of the integrals (2.10) and (1.1), we obtain, by eliminating x', y' and z', the phase trajectories as lines of intersection of the two phase surfaces

$$F_i(x, y, z, p, q, r) = F_i^*(x, y, z) = \text{const} \qquad (i = 1,2) \tag{2.11}$$

In order that the phase surfaces $F_i^*(x, y, z) = \text{const}$ $(i = 1, 2)$ have no common tangent planes, it is necessary that not all of the determinants

$$\frac{\partial(F_1^*, F_2^*)}{\partial(x, y)}, \qquad \frac{\partial(F_1^*, F_2^*)}{\partial(x, z)}, \qquad \frac{\partial(F_1^*, F_2^*)}{\partial(y, z)}$$

vanish simultaneously.

3. The self-contained system

$$x^{\cdot} = q - r, \qquad y^{\cdot} = r - p, \qquad z^{\cdot} = p - q \tag{3.1}$$

will be called conjugate to the basic system (1.1), and its phase trajectories, determined by the system of differential Eqs.

$$\frac{dx}{q-r} = \frac{dy}{r-p} = \frac{dz}{p-q} = dt \tag{3.2}$$

will be orthogonal to the phase trajectories of the basic system (1.1).

Denoting elementary displacements along the phase trajectories of the basic and conjugate system (1.1) and (3.1) respectively by $d\mathbf{s} = \mathbf{v}dt$ and $d\mathbf{s}^* = \mathbf{v}^*dt$, where $\mathbf{v} = p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ and $\mathbf{v}^* = (q - r)\mathbf{i} + (r - p)\mathbf{j} + (p - q)\mathbf{k}$ are the velocity of the phase points, we obtain

$$d\mathbf{s}\cdot d\mathbf{s}^* = (p\,(q - r) + q\,(r - p) + r\,(p - q))\,dt^2 = 0$$

which proves the orthogonality of the phase trajectories of the two systems.

The system of two Pfaffian equations obtained from (3.2) admit integral manifolds with two variables. This follows from the fact that $dx + dy + dz = 0$, and consequently the relation

$$x + y + z = C \qquad (C = \text{const}) \tag{3.3}$$

holds.

To effect further integration of (3.2) we use (3.3) to eliminate any one variable, let us say z. This leads to the integration of a Pfaffian Eq. of the form

$$A\,(x, y, C)\,dx + B\,(x, y, C)\,dy = 0 \tag{3.4}$$

$$A\,(x, y, C) = r\,(x, y, C - (x + y)) - p\,(x, y, C - (x + y))$$

$$B\,(x, y, C) = r\,(x, y, C-(x + y)) - q\,(x, y, C - (x + y))$$

Suppose $\Phi(x, y, C) = C_1$ is an integral of Eq. (3.4). Then the phase trajectory of the conjugate system (3.1) is a line of intersection of the cylindrical surface $\Phi(x, y, C) = C_1$ with the plane $x + y + z = C$.

Thus the problem of finding the phase trajectories of the basic system (1.1) reduces to the construction of lines orthogonal to the trajectories of the conjugate system (3.1).

4. We return now to system (1.1) and suppose that the functions $p(x, y, z)$, $q(x, y, z)$ and $r(x, y, z)$ are such that the condition (2.4) of conservativeness of the associated force field is now fulfilled. We seek a transformation of the form

$$dt = \omega\,(x, y, z)\,d\tau \tag{4.1}$$

which, while not changing the form of the phase trajectories, will make the transformed associated field conservative. Transformations similar to (4.1) were used in particular by Chaplygin [7] in the investigation of nonholonomic systems, and also by Birkhoff [5] in studying general Lagrangian systems.

In order that the force field, transformed by (4.1), be conservative, it is sufficient to require $\operatorname{rot}\mathbf{v}_1 = 0$, where $\mathbf{v}_1 = \omega\mathbf{v}$. This condition yields

$$(\operatorname{grad}\omega \times \mathbf{v}) + \omega \operatorname{rot}\mathbf{v} = 0$$

which may be reduced to the form

$$\operatorname{grad}\,\ln\omega \cdot \mathbf{v}^* = -\operatorname{div}\mathbf{v}^* \tag{4.2}$$

Here $\mathbf{v}^* = (q - r)\mathbf{i} + (r - p)\mathbf{j} + (p - q)\mathbf{k}$ is the previous velocity of the phase point of the conjugate system (3.1).

Hence the problem of finding a function $\omega(x, y, z)$, which, following Chaplygin, we shall call a reducing multiplier, reduces to the integration of the system

$$\frac{dx}{q-r}=\frac{dy}{r-p}=\frac{dz}{p-q}=\frac{d\ln\omega}{-\operatorname{div}\mathbf{v}^{\bullet}} \tag{4.3}$$

To integrate this system one may use the results given in the preceding Section.

5. We consider the simplest case, when the functions p, q and r in (1.1) are linear forms in the independent variables denoted by x_1, x_2, x_3. System (1.1) may then be written in vector notation as

$$\mathbf{x}^{\cdot}=A\mathbf{x}, \qquad \mathbf{x}=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \quad A=(a_{ij})_1^3 \tag{5.1}$$

Here $\mathbf{x}$ is a column vector and A a constant matrix of order three ($m = 3$).

Suppose the associated force field (2.1) is conservative and, consequently, satisfies condition (2.4). We consider the possible cases.

a) Assume $\Omega = \operatorname{rot}\mathbf{v} = 0$, so that the fluid motion connected with the system is a potential flow. By virtue of (5.1) this reduces to the condition $a_{ij} = a_{ji}$, so that the matrix A is symmetric. Furthermore the divergence of the velocity vector will equal the trace of the matrix A:

$$\operatorname{div}\mathbf{v}=\operatorname{tr}A=a_{11}+a_{22}+a_{33} \tag{5.2}$$

b) Suppose $\operatorname{rot}\mathbf{v}\parallel\mathbf{v}$. This case also has a simple hydrodynamical interpretation, namely that the vortex lines in the fluid flow are parallel to the phase trajectories.

Since for the system (5.1) $\operatorname{rot}\mathbf{v}$ is a constant vector, the vortex lines as well as the phase trajectories will be straight lines. This result may be obtained immediately by integrating the system

$$\frac{dx}{\xi_0}=\frac{dy}{\eta_0}=\frac{dz}{\zeta_0}=dt \qquad (\operatorname{rot}\mathbf{v}=\mathbf{\Omega}_0(\xi_0,\eta_0,\zeta_0))$$

c) Suppose $\operatorname{rot}\mathbf{v} = \text{const}$. Denoting $\operatorname{rot}\mathbf{v} = \Omega_0 (\Omega_0 \neq 0)$, we use (2.8) to obtain

$$\lambda\Omega_0=(\Omega_0\cdot\nabla)\mathbf{v} \qquad (\lambda=\operatorname{div}\mathbf{v}=a_{11}+a_{22}+a_{33}) \tag{5.3}$$

This relation together with (5.1) may be reduced to the form

$$(\mathbf{A}-\lambda\mathbf{E})\,\mathbf{\Omega}_0=0 \qquad \left(\mathbf{\Omega}_0=\begin{pmatrix}\xi_0\\\eta_0\\\zeta_0\end{pmatrix}\right) \tag{5.4}$$

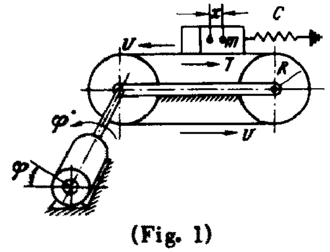
Here Ω_0 is a three-dimensional column-vector, and $\mathbf{E}$ is the identity matrix.

Since the vector Ω_0 is different from zero, condition (5.4) is equivalent to the characteristic Eq. $|A - \lambda E| = 0$ of the matrix A. Consequently $\lambda = \lambda_i$ ($i = 1, 2, 3$), equal to $\operatorname{div}\mathbf{v}$, are eigenvalues of the matrix A. We note that the matrix A will be singular in the case when $\operatorname{div}\mathbf{v} = 0$ ($\lambda = 0$).

6. As an example of a nonlinear system we examine a self-oscillatory system interacting with an energy source maintaining the oscillation (Fig. 1). We write the equations of motion in the following form [3]:

$$x^{\cdot\cdot}+2nx^{\cdot}+k^2x=aT(v-x^{\cdot}),$$

$$\varphi^{\cdot\cdot}+A\varphi^{\cdot}=BT(v-x^{\cdot}) \tag{6.1}$$



(Fig. 1)

Here x is the displacement of the oscillation mass m; ϕ is the angle of rotation of the rotor of an engine; $v = R\phi^{\cdot}$ is the velocity of the belt at the point of contact; $T(v - x^{\cdot})$ is the sliding friction force, depending on the relative velocity $v_r = v - x^{\cdot}$, and $k^2 = c/m$, n, a, A, and B are constants. We introduce the following phase variables: x, $y = v - x^{\cdot}$, and $z = \phi^{\cdot}$. As a result the system (6.1) will take the form

$$x^{\cdot} = p = Rz - y, \qquad y^{\cdot} = q = k^2x - 2ny + (2n - A)\ Rz + (BR - a)\ T(y) \qquad (6.2)$$
$$z^{\cdot} = r = -Az + BT(y)$$

We shall make no assumptions on the friction force $T(y)$. From physical considerations it follows that $T(y)$ should be an odd function. In the simplest case, when $T(y)$ is a linear function of y, the system (6.2) is also linear.

To integrate system (6.2) we use (3.3) and, under the simplifying assumption that $A + R = 0$, obtain for x a linear differential equation of first order.

Omitting the intermediate calculations, we write the final result

$$x = h(y)\left(C_1 + \int \frac{Q(y)}{h(y)}\, dy\right) \qquad \left(h = \exp \int \frac{M\, dy}{y + BT(y)},\ Q = \frac{M_1 + M_2 y + M_3 T(y)}{y + BT(y)}\right) \qquad (6.3)$$

Here C_1 is a constant of integration and

$$M = k^2 - 2nR + AR - A, \qquad M_1 = C\,(k^2 - M)$$
$$M_2 = M - k^2 - 2n, \qquad M_3 = BR - B - a$$

The intersections of the cylindrical surfaces $F(x, y, C_1) = 0$ (6.3) with the planes $x + y + z = C$ provides a system of lines orthogonal to the phase trajectories of the original system (6.2).

We define the reducing multiplier ω according to (4.3). Since the divergence of the velocity vector of the phase point for the conjugate system is:

$$\operatorname{div} \mathbf{v}^* = D + BT'(y) \qquad (D = 1 + k^2 + R\,(1 + A - 2n)) \qquad (6.4)$$

then by virtue of (4.3) we obtain

$$\omega(y) = \exp\left(-\int \frac{D + BT'(y)}{y + BT(y)}\, dy\right) \qquad (6.5)$$

This function assumes a particularly simple form when $D = 1$.

This yields $\omega = 1/(y + BT(y))$.

In conclusion we mention that the results obtained here may be generalized and extended to the case of n-dimensional self-contained systems.

BIBLIOGRAPHY

1. Liu Ruey – wen, Fett G.H., Analogy of nonlinear systems to classical dynamics. J. Franklin Inst., Vol. 272, No. 4, 1961.
2. Belen'kii, I.M., Applying the method of adjoint fields to nonlinear systems. PMM, Vol. 31, No. 2, 1967.
3. Kononenko, V.O., Oscillatory Systems with Bounded Perturbations. pp. 171-174, Izd. "Nauka" 1964.
4. Kochin, N.E., Kibel', I.A. and Roze, N.V., Theoretical Hydromechanics, Part 1. page 54, Fizmatgiz, 1963.
5. Birkhoff, J.D., Dynamical Systems. M.-L., OGIZ, pp. 38, 39 Russian translation, 1941.
6. Nemytskii, V.V., Topological classification of singular points and generalized Liapunov functions. Differential Equations. Minsk. Izd. "Nauka i Tekhnika", Vol. 3, No. 3, 1967.
7. Chaplygin, S.A., On the Theory of Motion of Nonholonomic Systems. Theorem on the Reducing Multiplier. Poln. sobr. soch. (Collected Works), Vol. 1, (pp. 208, 209), Izd. Akad. Nauk SSSR, 1933.

Translated by P.F.